Proof of the Sheldon Conjecture

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Abstract. In [**3**], the authors introduce the concept of a Sheldon prime, based on a conversation between several characters in the CBS television situation comedy *The Big Bang Theory*. The authors of [**3**] leave open the question of whether 73 is the unique Sheldon prime. This paper answers this question in the affirmative.

1. INTRODUCTION. A Sheldon prime was first defined in [**3**] as an homage to Sheldon Cooper, a fictional theoretical physicist, see Figure 1, on the television show *The Big Bang Theory*, who claimed 73 is the best number because it has some seemingly unusual properties. First note that not only is 73 a prime number, its index in the sequence of primes is the product of its digits, namely 21: it is the 21st prime. In addition, reversing the digits of 73, we obtain the prime 37, which is the 12th prime, and 12 is the reverse of 21.

We give a more formal definition. For a positive integer n, let p_n denote the nth prime number. We say p_n has the *product property* if the product of its base-10 digits is precisely n. For any positive integer x, we define $\mathrm{rev}(x)$ to be the integer whose sequence of base-10 digits is the reverse of the digits of x. For example, $\mathrm{rev}(1234) = 4321$ and $\mathrm{rev}(310) = 13$. We say p_n satisfies the *mirror property* if $\mathrm{rev}(p_n) = p_{\mathrm{rev}(n)}$.

Definition. The prime p_n is a Sheldon prime if it satisfies both the product property and the mirror property.

In [**3**], the "Sheldon Conjecture" was posed that 73 is the only Sheldon prime. In Section 5 we prove the following result.

Theorem 1. *The Sheldon conjecture holds:* 73 *is the unique Sheldon prime.*

2. THE PRIME NUMBER THEOREM AND SHELDON PRIMES. Let $\pi(x)$ denote the number of prime numbers in the interval $[2, x]$. Looking at tables of primes it appears that they tend to thin out, becoming rarer as one looks at larger numbers. This can be expressed rigorously by the claim that $\lim\limits_{x\to\infty} \pi(x)/x = 0$. In fact, more is true: we know the rate at which the ratio $\pi(x)/x$ tends to 0. This is the prime number theorem:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1,$$

where "log" is the natural logarithm function. This theorem was first proved in 1896 independently by Hadamard and de la Vallée Poussin, following a general plan laid out by Riemann about 40 years earlier (the same paper where he first enunciated the now famous Riemann hypothesis).

We actually know that $\pi(x)$ is slightly larger than $x/\log x$ for large values of x; in fact there is a secondary term $x/(\log x)^2$, a positive tertiary term, and so on. The phrase "large values of x" can be made numerically explicit: A result of Rosser and Schoenfeld [**7**, (3.5)] is that

$$\pi(x) > \frac{x}{\log x} \text{ for all } x \geq 17. \tag{1}$$

This beautiful inequality immediately allows us to prove that no Sheldon prime exceeds 10^{45}, and in fact, we only need the product property to show this.

Figure 1. Sheldon always knew 73 was the best.
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Proposition 2. *If p_n has the product property, then $p_n < 10^{45}$.*

Proof. Say p_n has k digits with the leading digit a. Then n, which is equal to the product of the digits of p_n, is at most $a\times 9^{k-1}$. Using (1), for $p_n \geq 17$, we have

$$n = \pi(p_n) > \frac{p_n}{\log p_n}.$$

But $p_n \geq a\times 10^{k-1}$ since p_n is k digits long. Thus if p_n has the product property, then the following inequality must be satisfied:

$$a\times 9^{k-1} > \frac{a\times 10^{k-1}}{\log(a\times 10^{k-1})},$$

which implies that

$$\log a + \log(10^{k-1}) > \left(\frac{10}{9}\right)^{k-1}. \tag{2}$$

Since the left side grows linearly in k and the right side grows exponentially, it is clear that (2) fails for all large values of k. Further, if (2) fails for $a = 9$, then it also fails for smaller values of a. A small computation and mathematical induction allow us to see that (2) fails for all $k \geq 46$. ■

Finer estimates than (1) exist in the literature, some of which are referenced below in Section 4. However, they do not afford much of an improvement in Proposition 2.

In [**3**], the authors show that the primes $p_7 = 17, p_{21} = 73$, and $p_{181{,}440} =$ 2,475,989 each satisfy the product property. This leads us to the following conjecture.

Conjecture 3. *The only primes with the product property are*

$$p_7 = 17, \quad p_{21} = 73, \quad \textit{and} \quad p_{181{,}440} = 2{,}475{,}989.$$

We have exhaustively searched for primes p_n with the product property for all $n \leq 10^{10}$ (using the built-in *Mathematica* function that gives the nth prime), and found only the 3 examples listed above. It is certainly possible to extend this search, but it seems computationally challenging to cover all of the territory up to 10^{45}.

An example of a challenging number to analyze is

$$n = 276{,}468{,}770{,}930{,}688 = 2^{17}3^{16}7^2.$$

It is not impossible, but difficult, to compute p_n. Short of this, if only we could approximate p_n we might be able to determine its most significant digits, which may allow us to rule it out. As discussed in Section 4 below, this approximation is afforded by the inverse function of the logarithmic integral function, namely $\mathrm{li}^{-1}(n)$. Definitions will be forthcoming, but for now note that Lemma 7 shows that

$$9{,}897{,}979{,}324{,}865{,}422 < p_n < 9{,}897{,}979{,}533{,}554{,}693. \tag{3}$$

We deduce that the top 7 digits of p_n are $9, 8, 9, 7, 9, 7, 9$, and the 8th digit must be a $3, 4$, or 5. The product of the first 7 digits is $2{,}571{,}912$, and the quotient after dividing this into n is $107{,}495{,}424 = 2^{14}3^8$. If p_n were to satisfy the product property, we see that the remaining 9 digits in p_n must consist of four 9's, four 8's, and one 4. Thus, we may assume the 8th digit of p_n is 4, and the last digit is 9. There are still 35 possibilities for the placement of the remaining digits. Although we may hope each would result in a composite number, that is not the case. For example, we have the candidates

$$9{,}897{,}979{,}489{,}888{,}999,$$
$$9{,}897{,}979{,}489{,}989{,}889,$$
$$9{,}897{,}979{,}489{,}998{,}889,$$
$$9{,}897{,}979{,}498{,}889{,}899.$$

Each of the above is prime, the product of their digits is n, and the only thing in doubt is their indices in the sequence of primes. These indices are all near n, but there are still many possibilities. It is certainly a tractable problem to find these indices, but it seems there will be many similar and much harder challenges as one searches higher.

To prove our theorem that 73 is the only Sheldon prime we will make use of the mirror property in addition to the product property. For example, the reverses of the 4 primes above are all composite, so they are instantly ruled out as Sheldon primes.

Though the bound 10^{45} may seem daunting, we see at least that the search for Sheldon primes is finite. Our basic strategy is to use numerically explicit versions of the prime number theorem, similar to, but finer than, (1) to give us some of the leading and trailing digits of candidate primes, and use these to, we hope, eliminate them. Further, our search is not over all primes to 10^{45} but over integers n with $p_n < 10^{45}$.

But first we need to assemble our weapons for the attack!

3. PROPERTIES OF SHELDON PRIMES. Because a Sheldon prime must satisfy both the product property and the mirror property (described in the Introduction), there are a few simple tests one can apply to candidates based on properties of Sheldon primes.

Proposition 4. *If p_n is a Sheldon prime and $n > 10^{10}$, then*

1. *n is 7-smooth (meaning that no prime dividing n exceeds 7);*
2. *the leading digit of p_n must be in $\{1, 3, 7, 9\}$;*
3. *the number of digits of $p_{\mathrm{rev}(n)}$ must equal the number of digits of p_n;*
4. *$5^4 \nmid n$;*
5. *if $p_n > 10^{19}$, then $5^3 \nmid n$;*
6. *$100 \nmid n$;*
7. *p_n cannot have a digit 0, and cannot have a digit 1 except possibly for the leading digit;*
8. *the leading digit of $p_{\mathrm{rev}(n)}$ must be in $\{3, 7, 9\}$.*

Proof. Part (1) is immediate from the definition of the product property. Parts (2) and (3) are clear since $\mathrm{rev}(p_n)$ must be prime, and primes beyond single digits must end in $1, 3, 7$, or 9. Noting that each factor 5 in n must come from a digit 5 in p_n, one can prove part (4) using the same method as the proof of Proposition 2. In particular,

$$a \times 5^4 \times 9^{k-5} < \frac{a \times 10^{k-1}}{\log(a \times 10^{k-1})}$$

for $a = 1, 3, 7$, or 9 and $k \geq 5$. One can similarly derive part (5).

For part (6), we direct the reader to [**3**], where a detailed proof is given. The idea is that if $100 \mid n$, then $\mathrm{rev}(n) < \frac{1}{10}n$, while $\mathrm{rev}(p_n)$ has the same number of digits as p_n. Prime number inequalities, such as (6), complete the proof.

It is obvious that no prime having the product property can have a digit 0. For the second part of (7), suppose that p_n has a digit 1 after the leading digit. But

$$a \times 9^{k-2} < \frac{a \times 10^{k-1}}{\log(a \times 10^{k-1})}$$

for $a = 1, 3, 7$, or 9 and $k \geq 6$. This proves (7), and since we now know that the trailing digit of p_n cannot be 1, part (8) follows immediately. ■

With Proposition 4 in hand, we are almost ready to begin the search to 10^{45}. However, it is not so simple to compute p_n for large numbers n. What is simple is computing the inverse of the logarithmic integral function, $\mathrm{li}^{-1}(n)$, and so we would like to know how close this is to p_n. The tools in the next section give us some guidance in this regard.

4. BOUNDS. We will make use of the first Chebyshev function, $\theta(x) = \sum_{p \leq x} \log p$, where p runs over prime numbers. We will also require $\mathrm{li}(x) = \int_0^x \frac{dt}{\log t}$, the logarithmic integral function. Here the "principal value" is taken at the singularity at $t = 1$; that is, if $x > 1$, then

$$\mathrm{li}(x) = \lim_{y \to 0^+} \left(\int_0^{1-y} \frac{dt}{\log t} + \int_{1+y}^x \frac{dt}{\log t} \right).$$

This is the traditional way of defining $\mathrm{li}(x)$ and it has its advantages, but it admittedly makes the function $\mathrm{li}(x)$ look very complicated, and doing so only adds a constant to the perhaps more natural $\displaystyle\int_2^x \frac{dt}{\log t}$. The function $\mathrm{li}(x)$ is a much better approximation to $\pi(x)$ than is $x/\log x$ and it is why we introduce it. In any event, $\mathrm{li}(x)$ is asymptotic to $x/\log x$ as $x \to \infty$, in that

$$\lim_{x\to\infty} \frac{\mathrm{li}(x)}{x/\log x} = 1.$$

(This can be easily proved using L'Hopital's rule.) We shall also be using the inverse of $\mathrm{li}(x)$, namely $\mathrm{li}^{-1}(x)$, which satisfies

$$\lim_{x\to\infty} \frac{\mathrm{li}^{-1}(x)}{x\log x} = 1.$$

It is of interest to us because $\mathrm{li}^{-1}(n)$ is a very good approximation to p_n. How good we shall see shortly.

Let

$$\begin{aligned} A(x) &= x - \theta(x), \\ B(x) &= \mathrm{li}(x) - \pi(x), \\ C(n) &= p_n - \mathrm{li}^{-1}(n). \end{aligned}$$

We wish to find numerically explicit bounds for $|C(n)|$. Lemma 5 relates the functions A and B, and Lemma 6 relates B and C. We will use these relations to ultimately obtain bounds for $|C(n)|$.

Lemma 5. *For $x > a > 2$, we have*

$$B(x) - B(a) = \frac{A(x)}{\log x} - \frac{A(a)}{\log a} + \int_a^x \frac{A(t)}{t(\log t)^2}\,dt.$$

Proof. This result follows from "partial summation," a discrete analogue of integration by parts. However, we may verify the identity directly. Note that

$$\int_a^x \frac{dt}{t(\log t)^2} = \frac{1}{\log a} - \frac{1}{\log x},$$

so that

$$\begin{aligned} \int_a^x \frac{\theta(t)}{t(\log t)^2}\,dt &= \int_a^x \frac{\theta(a)}{t(\log t)^2}\,dt + \sum_{a<p\le x} \int_p^x \frac{\log p}{t(\log t)^2}\,dt \\ &= \frac{\theta(a)}{\log a} - \frac{\theta(a)}{\log x} + \sum_{a<p\le x} \left(\frac{\log p}{\log p} - \frac{\log p}{\log x} \right) \\ &= \frac{\theta(a)}{\log a} - \frac{\theta(x)}{\log x} + \pi(x) - \pi(a). \end{aligned}$$

Since

$$\int_a^x \frac{dt}{(\log t)^2} = \mathrm{li}(x) - \mathrm{li}(a) - \frac{x}{\log x} + \frac{a}{\log a},$$

we thus have

$$\int_a^x \frac{A(t)}{t(\log t)^2}\,dt = \int_a^x \frac{t-\theta(t)}{t(\log t)^2}\,dt = B(x) - B(a) - \frac{A(x)}{\log x} + \frac{A(a)}{\log a},$$

and the result is proved. ∎

We will choose some convenient number for a where $A(a)$, $B(a)$ have been computed ($a = 10^{19}$ in Proposition 10).

Lemma 6. *For any integer $n > 0$, we have*

$$|C(n)| \le |B(p_n)| \log\left(\max\{p_n, \mathrm{li}^{-1}(n)\}\right).$$

Proof. We apply the mean value theorem to the function li on the interval with endpoints p_n and $\mathrm{li}^{-1}(n)$ to obtain

$$\mathrm{li}(p_n) - n = \frac{p_n - \mathrm{li}^{-1}(n)}{\log u},$$

for some value of u in the interval. Thus,

$$C(n) = B(p_n)\log u, \tag{4}$$

and taking absolute values, the result follows. ∎

We will split the positive integers into two intervals: those at most 10^{19} and those above 10^{19}. If we are in the lower range, then Büthe [**2**, Theorem 2] gives the following strong inequality. For $2 \le x \le 10^{19}$,

$$0 < B(x) < \frac{\sqrt{x}}{\log x}\left(1.95 + \frac{3.9}{\log x} + \frac{19.5}{(\log x)^2}\right). \tag{5}$$

This allows us to use (4) to obtain the following bound on $C(n)$.

Lemma 7. *For $p_n < 10^{19}$,*

$$0 < C(n) < \sqrt{p_n}\left(1.95 + \frac{3.9}{\log p_n} + \frac{19.5}{(\log p_n)^2}\right).$$

Proof. This follows immediately from (4) and (5) provided $\log u \le \log p_n$, where u is in the interval with endpoints p_n and $\mathrm{li}^{-1}(n)$. But (5) implies that $\mathrm{li}^{-1}(n) < p_n$ when $p_n < 10^{19}$. ∎

Note that if we only know n and are not sure what p_n is we can still use Lemma 7 if we combine it with the simple upper bound from [**7**, (3.13)]:

$$p_n < n(\log n + \log\log n), \quad n \ge 6. \tag{6}$$

For example, we have (3) from Section 2. For another example, suppose $n = 3^{35}$. We compute that

$$\mathrm{li}^{-1}(3^{35}) = 2.05844182653518213541\times 10^{18},$$

with an error smaller than 0.01. The error bound given by (6) and Lemma 7 is less than 3×10^9. Thus p_n has 19 digits and the leading 9 of them are 205844182. This p_n is obviously not a Sheldon prime, as it will clearly fail the product property.

To complement our upper bound (6) we shall need the following lower bound for $\mathrm{li}^{-1}(x)$.

Lemma 8. *For $x \geq 12{,}218$ we have*

$$\mathrm{li}^{-1}(x) > x(\log x + \log\log x - 1).$$

Proof. This inequality is clearly equivalent (since li is an increasing function) to

$$\mathrm{li}(x(\log x + \log\log x - 1)) < x$$

for $x \geq 12{,}218$. Note that it holds at $x = 12{,}218$. So, it will follow if we show that

$$\frac{d}{dx}(\mathrm{li}(x(\log x + \log\log x - 1)) < 1 \tag{7}$$

in the same range. The derivative in (7) is

$$\frac{\log x + \log\log x + 1/\log x}{\log x + \log(\log x + \log\log x - 1)}.$$

Letting $z = \log x$, this derivative is

$$\frac{z + \log z + 1/z}{z + \log(z + \log z - 1)} = 1 - \frac{\log(1 + (\log z - 1)/z) - 1/z}{z + \log(z + \log z - 1)}.$$

We thus would like to show this last numerator is positive. Using the inequality $\log(1 + w) > w/(1 + w)$ for $w > 0$, the numerator is larger than

$$\frac{\log z - 1}{z + \log z - 1} - \frac{1}{z}.$$

It's clear then that this is positive for large enough values of z, and we check in fact that $z \geq 8.46$ is sufficient. This holds if $x \geq 5000$, so we have shown (7) and thus the lemma. ∎

For $x \geq 10^{19}$, we use the following estimate for $|A(x)|$ from [**5**, Proposition 2.1] that uses bounds of Büthe [**1**].

Lemma 9. *For $x \geq 10^{19}$, $|A(x)| < \epsilon x$, with $\epsilon = 2.3\times10^{-8}$.*

With Lemma 9, we can now construct our remaining upper bound for $C(n)$.

Proposition 10. *Let*

$$E(x) = \left(5.5\times10^9 + 2.3\times10^{-8}\mathrm{li}(x) + 10^{-11}x\right)\log x.$$

For $p_n > 10^{19}$, we have

$$|C(n)| < E\left(\mathrm{li}^{-1}(n)\right).$$

Proof. Let $a = 10^{19}$. Using (5), we have $|B(a)| < 2\times10^8$. We now use Lemma 9 and $\frac{d}{dt}(\text{li}(t) - t/\log t) = 1/(\log t)^2$ to get for $x > a$,

$$\begin{aligned}
\frac{|A(x)|}{\log x} &< 2.3\times10^{-8}\frac{x}{\log x},\\
\frac{|A(a)|}{\log a} &< 5.3\times10^9, \text{ and}\\
\int_a^x \frac{|A(t)|}{t(\log t)^2}\,dt &< 2.3\times10^{-8}\int_a^x \frac{dt}{(\log t)^2} < 2.3\times10^{-8}\left(\text{li}(x) - \frac{x}{\log x}\right).
\end{aligned}$$

We thus conclude from Lemma 5 that for $x > 10^{19}$,

$$\begin{aligned}
|B(x)| &\le |B(a)| + \frac{|A(a)|}{\log a} + \frac{|A(x)|}{\log x} + \int_a^x \frac{|A(t)|}{t(\log t)^2}\,dt\\
&< 5.5\times10^9 + 2.3\times10^{-8}\text{li}(x).
\end{aligned}$$

Let

$$E_1(x) = \left(5.5\times10^9 + 2.3\times10^{-8}\text{li}(x)\right)\log x,$$

so that from Lemma 6 we have for $p_n > 10^{19}$ that

$$|C(n)| \le E_1\left(\max\{p_n, \text{li}^{-1}(n)\}\right).$$

The proposition follows in the case that $p_n \le \text{li}^{-1}(n)$. Suppose the reverse inequality holds, that is, $p_n > \text{li}^{-1}(n)$. We use the upper bound (that's evidently an improvement on (6)!)

$$p_n < n\Big(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n}\Big), \quad n \ge 688{,}383,$$

found in [**4**, Proposition 5.15]. With this and also using Lemma 8, we find that

$$E_1(p_n) < E(\text{li}^{-1}(n))$$

for $n > 10^{17}$, a range which includes $p_n > 10^{19}$ (using Lemma 7), so completing the proof. ∎

Our calculations were performed using *Mathematica*. In particular, we used the built-in function `LogIntegral[x]` for $\text{li}(x)$. Starting from the approximation $x(\log x + \log\log x - 1)$, we were then able to use a few iterations of Newton's method to compute $\text{li}^{-1}(x)$ for numbers x of interest to us.

5. PROOF OF THEOREM 1. We first search over any primes less than 10^{19}. By Lemma 7, if $p_n < 10^{19}$ then $n \le N := 2.341\times10^{17}$. So we begin our search by creating a list of all 7-smooth numbers up to N. This is quickly computed by creating a list of numbers of the form $2^a3^b5^c7^d$, with

$$\begin{aligned}
0 &\le a \le \log_2(N),\\
0 &\le b \le \log_3(N/2^a),
\end{aligned}$$

$$0 \le c \le \log_5(N/(2^a 3^b)),$$
$$0 \le d \le \log_7(N/(2^a 3^b 5^c)).$$

In particular, there are $57{,}776$ integers of this form. We remove the $7{,}575$ members of the table that are at most 10^{10} since we have previously searched over these numbers n to recover the 3 primes satisfying the product property.

We will use the properties of Proposition 4 to eliminate the remaining $50{,}201$ values of n. First we remove those n where $100 \mid n$ or $5^4 \mid n$, leaving $13{,}335$ numbers. For these n, we compute $\mathrm{li}^{-1}(n)$ and check via Lemma 7 that the leading digit is well determined. Similarly we check that the number of digits of p_n is well determined. Then we select those with the leading digit of p_n in $\{1, 3, 7, 9\}$. These filters reduce our list to a possible $6{,}893$ candidates.

We then select those n where the top 5 digits of p_n are given by the top 5 digits of $\mathrm{li}^{-1}(n)$. All but 68 values of n have this property. Using the same method as Proposition 2, we assume that all the remaining digits of p_n are 9's and check to see if the product of these 9's and the top 5 known digits is at least n. If not, we can rule out n, and this eliminates all but 576 cases. For these cases, we check if the top 6 digits are given by $\mathrm{li}^{-1}(n)$, and all but 61 of them still have this property. We then repeat the product test with the top 6 digits and this leaves only 180 numbers. Combined, our three remaining sets together total $309 = 68 + 61 + 180$ possible candidates.

For these remaining numbers n, we compute $\mathrm{rev}(n)$. By part (8) of Proposition 4, we reduce to the 60 of them with first digit of $p_{\mathrm{rev}(n)}$ in $\{3, 7, 9\}$. Of these, 55 of them have known top 5 digits. The 5 exceptions correspond to $\mathrm{rev}(n)$ being one of

$$4{,}019{,}155{,}056,\ 4{,}032{,}803{,}241,\ 4{,}079{,}545{,}092,$$
$$12{,}427{,}422{,}237,\ 29{,}794{,}252{,}274.$$

These are all small enough so that we can find the corresponding primes directly:

$$97{,}496{,}326{,}163,\ 97{,}841{,}660{,}857,\ 99{,}024{,}780{,}191,$$
$$316{,}109{,}730{,}941,\ 785{,}009{,}387{,}557.$$

They all have a digit 0 except for the first one, and that has an internal digit 1, and so these 5 are ruled out by part (7) of Proposition 4.

With the remaining 55 numbers, we can again use the product test with $\mathrm{rev}(n)$, as described above, and this eliminates all but 6 of them. These are too large to find the corresponding primes, but we can easily find how many digits the corresponding primes have, and only 2 of the 6 have the same number of digits as the primes corresponding to n. For these two, we know the leading 6 digits of p_n and the leading 5 digits of $p_{\mathrm{rev}(n)}$, which would need to be the trailing 5 digits of p_n if p_n were indeed a Sheldon prime. The product of these 11 digits times the appropriate power of 9 for the still-unknown digits is too small for these to be Sheldon primes. This completes the search up to 10^{19}.

For the remainder of the proof, we use Proposition 10. If $p_n < 10^{45}$, then $n < 9.746\times10^{42}$. We compute the 7-smooths to this bound; there are $1{,}865{,}251$ of them. Removing those less than 2.34×10^{17} and those divisible by 100 or 125 leaves a list of 213,449 remaining numbers. Each of these gives an unambiguous first digit for p_n, and then selecting those where the first digit is in $\{1, 3, 7, 9\}$ leaves $112{,}344$. We then verify that for each of these we can use $\mathrm{li}^{-1}(n)$ to determine the exact number of digits of p_n.

We then test if the first 5 digits of p_n are unambiguous and all but 168 of them have this property. For those that do have the property, we multiply the top 5 digits by an appropriate power of 9 to get an upper bound on the product of the digits of p_n, keeping only the 991 of them where this upper bound is at least n. We then repeat this procedure with the top 6 digits. All but 29 of them have the top 6 digits determined, and of the remaining values of n, all but 277 of them are discarded because the product of digits is too small. We then keep only those where the product of the first 6 digits divides n; there are 141 left.

We thus have a remaining set of size $338 = 168 + 29 + 141$ numbers n. For these, we check that the number of digits and the first digit of $p_{\mathrm{rev}(n)}$ is determined from $\mathrm{li}^{-1}(\mathrm{rev}(n))$. We then discard those where the number of digits of p_n is not equal to the number of digits of $p_{\mathrm{rev}(n)}$ and those where the top digit of $p_{\mathrm{rev}(n)}$ is not in $\{3, 7, 9\}$. This leaves only 45 numbers. Each of these has $\mathrm{li}^{-1}(\mathrm{rev}(n))$ able to determine the top 5 digits of $p_{\mathrm{rev}(n)}$, and all of these values of n fail the test where we multiply the top 5 digits of $p_{\mathrm{rev}(n)}$ and the appropriate power of 9 and check that against n.

This completes the proof that 73 is the unique Sheldon prime.

6. FUTURE WORK. Several generalizations and extensions of this concept naturally emerge from the above discussion. For instance, the product property of a Sheldon prime clearly rests on its base-10 representation. Can you classify all primes satisfying the product property in different bases? For instance, $226{,}697$ is the $20{,}160$th prime, and its base-9 representation is 374865_9. Multiplying its base-9 digits together returns $20{,}160$ and so we can say $226{,}697$ satisfies the product property in base-9.

Is there a meaningful way to describe a prime which nearly has the product property? For instance, $p_{35} = 149$. The product of the digits of 149 is 36, which is only 1 away from 35, and hence 149 is quite close to having the product property.

For a positive integer n, let $f(n)$ denote the product of the base-10 digits of p_n. Then an index n for which p_n has the product property satisfies $f(n) = n$, and conversely. If we iterate the function f we can find some longer cycles. For example, $f(1) = 2$, $f(2) = 3$, $f(3) = 5$, $f(5) = 1$. Since a cycle must contain a number n such that $f(n) \geq n$, it's clear from Proposition 2 that there are only finitely many cycles. Can one find any others? Note that an iteration comes to an end as soon as a number n is encountered such that p_n has a 0 digit. Otherwise an orbit eventually enters a cycle.

It is interesting to note that most primes do have a digit 0 in their decimal expansion, since the number of integers in $[2, x]$ with no digit 0 is at most about $x^{0.954}$ which is small compared with $\pi(x)$ (which we have seen is about $x/\log x$). One might guess there are infinitely many primes missing the digit 0, and in fact, this was recently proved by Maynard [**6**].

We showed in Proposition 2 that there are at most finitely many primes with the product property. What about the mirror property? In addition to 73, the primes 2, 3, 5, 7, and 11 all have the mirror property, which is easily verifed. They are all examples of "palindromic mirror primes" in that both p_n and n are palindromes. A larger example, from [**3**], is

$$p_{8{,}114{,}118} = 143{,}787{,}341.$$

A heuristic argument suggests that there are infinitely many primes with the mirror property, but the palindromic mirror primes occur more frequently than the mirror property primes that are not palindromes. In fact, up to x there should be about $\sqrt{\log x}$ palindromic mirror primes, and about $\log\log x$ mirror primes that are not palindromes.

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